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A SIMPLE EXISTENCE CRITERION FOR $(g<f)$-FACTORS*

Katherine HEINRICH, Pavol HELL
Department of Mathematics and Statistics, Simon Fraser University, Burnaby, B.C., Canada

David G. KIRKPATRICK
Department of Computer Science, University of British Columbia, Vancouver, B.C., Canada

Guizhen LIU
Department of Mathematics and Statistics, Simon Fraser University, Burnaby, B.C., Canada
Department of Mathematics, Shandong University, Jinan, Shandong Province, People's Rep. of China

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We simplify the criterion of Lovász for the existence of a (g, f)-factor when $g<f$, or when the graph is bipartite. Moreover, we give a simple direct proof, implying an $O(\sqrt{(g(V))}\cdot|E|)$ algorithm, for these cases. We then illustrate the convenience of the new criterion by deriving some old and some new facts about (g,f)-factors and $[a,b]$-factors.

Let G be a graph and let f and g be integer functions on the vertex set of G such that $0\leqslant g(x)\leqslant f(x)\leqslant \deg_G(x)$ for each vertex x of G. A (g,f)*-factor* of G is a spanning subgraph F of G with the property that $g(x)\leqslant \deg_F(x)\leqslant f(x)$ for each vertex x of G. When $g=a$ and $f=b$ are constants, it is customary to write square brackets; that is, we have an $[a,b]$-factor. If f is any real function on a set S, we let $f(S)=\sum_{x\in S}f(x)$. Lovász characterized graphs which admit a (g,f)-factor. His criterion is the following (cf. [11, 12]):

Theorem L. *The graph G has a (g,f)-factor if and only if for every pair of disjoint sets S and T,*

$$\varepsilon(S, T)+\tau(S, T)-\deg_G(T)\leqslant f(S)-g(T).$$

(Here $\varepsilon(S, T)$ denotes the number of edges between S and T, and $\tau(S, T)$ denotes the number of components C of $G-(S\cup T)$ such that $\varepsilon(T, V(C))+f(V(C))$ is odd and $f(x)=g(x)$ for all $x\in V(C)$. The expression $\deg_G(T)$ is defined by our general convention as the sum of $\deg_G(x)$ over the set T.)

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Compare this criterion with the well known characterization of Tutte [14] of graphs with a 1-factor: The graph G has a $[1,1]$-factor if and only if for every set S the number of odd components of $G-S$ is at most $|S|$. Tutte's criterion depends on only one set S rather than the pair of sets S, T of the above theorem. LasVergnas generalized Tutte's theorem to give a one-set criterion for the existence of (g,f)-factors in the special case when $g \leqslant 1$, [10]:

Theorem LV. *Assume that $g(x) \leqslant 1$ for each vertex x of G. Then G has a (g,f)-factor if and only if for every set S,*

$$g(I_S) + \sigma(S) \leqslant f(S).$$

(Here I_S is the set of isolated vertices of $G-S$ and $\sigma(S)$ denotes the number of other odd components C of $G-S$ such that $f(x)=g(x)=1$ for all $x \in V(C)$.)

We give below a one-set criterion for the existence of (g,f)-factors for the special case when $g<f$. The same criterion also applies when the graph G is bipartite. It is not difficult to derive these results directly from Lovász' theorem as stated above, or from the simpler (but still two-set) formulations for the cases $g<f$ or G bipartite in [12, 10.2.27–28]. However, the situation allows for a much simpler direct proof, from which one can also extract an efficient algorithm.

Theorem 1. *Assume that one of the following two conditions holds:*
(i) *for every vertex x of G we have $g(x)<f(x)$ or* (ii) *the graph G is bipartite. Then G has a (g,f)-factor if and only if for every set S*

$$\sum_{x \notin S} (g(x) \dot{-} \deg_{G-S}(x)) \leqslant f(S).$$

(Here '$\dot{-}$' denotes the 'monus' function defined for real numbers a, b by $a \dot{-} b = \max(0, a-b)$.)

Proof. As mentioned above, this can be deduced from Lovász' theorem. However, the general proofs of Lovász' theorem [11, 12, 16, 6] are quite complex. We present a very simple alternating-path proof. First of all, if G has a (g,f)-factor F, then for any set S and any $x \notin S$ for which $\deg_{G-S}(x) < g(x)$, there must be in F at least $g(x) - \deg_{G-S}(x)$ edges from x to S. Thus $\sum_{x \notin S} (g(x) \dot{-} \deg_{G-S}(x))$ is at most the number of edges from $G-S$ to S, which is at most $f(S)$. Thus the necessity of the condition follows. To prove the sufficiency, assume that G has no (g,f)-factor, and let F be a $(0,f)$-factor which minimizes the quantity

$$\delta(F) = \sum_{x \in V(F)} (g(x) \dot{-} \deg_F(x)).$$

(Note that the empty graph is a $(0,f)$-factor of G) Let U denote the set of vertices $x \in V(G)$ for which $\deg_F(x) < g(x)$. By minimality of $\delta(F)$ we can assume that each edge joining two vertices of U belongs to F. We also assume that $\delta(F)>0$. A u-alternating path with respect to F is a sequence of vertices of G, $u_1, u_2, \ldots, u_k = u$ such that $u_1 \in U$, and, for each i, $u_{2i}u_{2i+1} \in E(F)$, $u_{2i-1}u_{2i} \notin E(F)$; the path is even respectively odd if k is even respectively odd. Let S, respectively T, be the set of vertices $u \notin U$ for which there exists an even, respectively odd, u-alternating path. It follows from the minimality of $\delta(F)$, that for $x \in S$, $\deg_F(x) = f(x)$ and for $x \in T$, $\deg_F(x) = g(x)$. (If, say, k were even and $\deg_F(u_k) < f(u_k)$ then the symmetric difference of F and the alternating path $u_1, u_2, \ldots, u_k$ would be a $(0,f)$-factor with smaller δ.) It also follows from these definitions that if xy is an edge of F, then $x \in S$ implies $y \in T \cup U$, and if xy is an edge of G but not of F then $x \in T \cup U$ implies $y \in S$. Note that either of our assumptions, (i) or (ii), guarantees that S and T are disjoint. Thus, for $x \in T \cup U$,

$$\deg_{G-S}(x) = \deg_F(x) - d_{F,S}(x) \leq g(x),$$

where $d_{F,S}(x)$ is the number of edges of F joining x to S. Therefore,

$$\begin{aligned}\sum_{x \notin S} (g(x) \dot{-} \deg_{G-S}(x)) &\geq \sum_{x \in T \cup U} (g(x) \dot{-} \deg_{G-S}(x)) \\ &= \sum_{x \in U} (g(x) - \deg_F(x)) + \sum_{x \in T \cup U} d_{F,S}(x) > f(S),\end{aligned}$$

since $\sum_{x \in T \cup U} d_{F,S}(x) = f(S)$ and U is non-empty. □

The same proof can also be seen to imply that the minimum $\delta(F)$ of a $(0,f)$-factor F satisfies the following minmax relation (cf. [5]):

Corollary 1. $\min_F \delta(F) = \max_S(\sum_{x \notin S}(g(x) \dot{-} \deg_{G-S}(x)) - f(S))$.

Moreover, if we define an augmenting path in G with respect to a $(0,f)$-factor F to be either an even u-alternating path with $\deg_F(u) < f(u)$, or an odd u-alternating path with $\deg_F(u) > g(u)$, then the above proof shows that F minimizes $\delta(F)$ if and only if it does not admit an augmenting path. Since each search for an augmenting path can be performed by breadth first search in time $O(|E|)$ and the corresponding augmentation lowers the value $g(x) \dot{-} \deg_{G-S}(x)$ for at least one vertex x, we have a very simple (g,f)-factor algorithm of time complexity $O(g(V) \cdot |E|)$. Using a reduction to flows [6], or to matchings [4], we find $O(\sqrt{f(V)}) \cdot |E|)$ algorithms. In [7], we give $O(\sqrt{g(V)}) \cdot |E|)$ algorithms based on a direct generalization of [8]. (By a more complex route we also give $O(\sqrt{g(V)}) \cdot |E|)$ algorithms for the general (g,f)-factor problem, [7].)

We also have a common generalization to Theorems LV and 1; a one-set condition is given in [6] for the case where no vertex x has $f(x) = g(x) \geq 2$. (The condition differs from that given in Theorem 1 only in having the left-hand side increased by the quantity $\sigma(S)$.) Further generalizations and one-set criteria are given in [2].

Corollary 2. *Let g and f be as in Theorem* 1. *If for all pairs of vertices x, y of G,*

$$g(y)/\deg_G(y) \leqslant f(x)/\deg_G(x),$$

then G has a (g,f)-factor.

Proof. The first observation to make is that for any set S,

$$\sum_{x \notin S} (\deg_G(x) - \deg_{G-S}(x)) \leqslant \deg_G(S).$$

Then, letting $W = \{x \notin S\colon\ g(x) \geqslant \deg_{G-S}(x)\}$, we verify the condition of Theorem 1 as follows:

$$\begin{aligned}
&f(S) - \sum_{x \notin S} (g(x) - \deg_{G-S}(x)) \\
&\geqslant (f(S)/\deg_G(S)) \sum_{x \notin S} (\deg_G(x) - \deg_{G-S}(x)) - \sum_{x \notin S} (g(x) - \deg_{G-S}(x)) \\
&= (1/\deg_G(S)) \Bigg[\sum_{x \in W} ((f(S)\deg_G(x) - \deg_G(S) g(x)) \\
&\quad + (\deg_G(S) - f(S)) \deg_{G-S}(x)) \\
&\quad + f(S) \sum_{x \notin S \cup W} (\deg_G(x) - \deg_{G-S}(x)) \Bigg] \geqslant 0. \qquad \square
\end{aligned}$$

We now turn to $[a,b]$-factors.

Corollary 3. *Let* $1 \leqslant a < b$ *and let* $n(S, j)$ *denote the number of vertices of* $G - S$ *of degree j. Then G has an* $[a,b]$*-factor if and only if for any set S*

$$\sum_{0 \leqslant j < a} (a - j) n(S, j) \leqslant b \cdot |S|.$$

Corollary 3 generalizes an earlier result of Berge and Las Vergnas [3] for $[1,b]$-factors, $b \geqslant 2$. (Since $a = 1$, the left-hand side reduces to $|I_S|$.) Theorem 1 also implies further extensions. For instance, if $1 \leqslant a < b$, and if T is a given set of vertices of G, then G has a $[0,b]$-factor F with $a \leqslant \deg_F(x) \leqslant b$ for all x in T, if and only if $\sum_{0 \leqslant j < a} (a - j) n(S, T, j) \leqslant b \cdot |S|$, where $n(S, T, j)$ denotes the set of vertices of T having degree j in $G - S$. Corollary 2 generalizes the following result of Kano and Saito [9]: if $a < b \leqslant n$ and $a/b \leqslant m/n$, then every $[m,n]$-graph has an $[a,b]$-factor. (An $[m,n]$-graph has all degrees between m and n.) This was first proved by Tutte [15] for $n = m$ and $b = a + 1$, and by Thomassen [13] for $n = m + 1$ and $b = a + 1$. By repeated application of these results we obtain the following:

Corollary 4. *An $[m,n]$-graph has k edge-disjoint $[a,b]$-factors if*

$$mb\text{-}na \geq (k-1)(b^2-a^2) \quad \text{and} \quad a<b\leq n-(k-1)a.$$

So, in particular, under the conditions $r\geq(k-1)(a+b)$ and $a<b\leq r-(k-1)a$ every r-regular graph has k disjoint $[a,b]$-factors.

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